International Journal of Theoretical Physics, Vol. 28, No. 6, 1989

New Class of Finsler Metrics

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Received January 1, 1989

A class of Finsler spaces is introduced which is determined by the metric function $F(x, y) = [(\eta_{\alpha\beta} + kB_\alpha B_\beta)y^\alpha y^\beta]^{1/2}$, where $B_\alpha = B_\alpha(x, y)$ and k is a constant. Various properties of these spaces are developed. A particular choice of B_α is shown to produce a geodesic equation which is equivalent to the Lorentz equation of motion for a charged particle. Some general arguments for the physical applicability of Finsler spaces are also given.

1. INTRODUCTION

There is increasing interest in Finsler spaces as possible generalizations of the traditional Riemannian approach to general relativity. A number of recent papers (see, e.g., Asanov, 1977; Ikeda, 1985; Aringazin and Asanov, 1985; Tavakol and Van den Bergh, 1986) have described particular examples of Finsler spaces of potential physical significantce. The reader unfamiliar with Finsler space should see the book by Asanov (1985), which is a valuable addition to the standard reference of Rund (1959).

The application of Finsler spaces has been inhibited by some long-standing physical objections (Alder *et al.*, 1975; and Asanov, 1985). These objections might be summarized as follows:

(i) Light cones may not be unique and may not correspond to null geodesics.

(ii) The norm of vectors may not be constant under transport along certain paths.

Corollaries of (ii) are that timelike (or spacelike) vectors might not remain timelike (or spacelike) and that vectors orthogonal at one point in space might not be orthogonal after being transported to other points.

Work of the past few years, however, has pointed out that there are classes of Finsler space which can overcome objections (i) and (ii). Asanov

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0020-7748/89/0600-0659$06.00/0

(1985) discusses spaces of the one-form or Berwald–Moór type, while Tavakol and Van den Bergh (1986) describe specific Berwald spaces with a conformal multiplying factor.

There are also some comprehensive arguments which can be applied to counter (i) and (ii) for Finsler spaces in general. These arguments will be presented briefly in Section 4. For the time being it is assumed that Finsler spaces can indeed be applied to some physical problems. The proof of this will be that equations derived from Finsler analysis compare with equations that are known to be pertinent to those problems.

In the present work a new class of Finsler space is discussed. A simple example of one of these spaces was introduced in Beil (1987).

In Section 2 the general form of this new class of spaces is given and some of its properties are derived. In particular, a general geodesic equation is developed. In Section 3 some special examples of these metrics are examined and one is selected for detailed computation. The Finsler geodesic equation for this metric is shown to be equivalent to the Lorentz equation for a charged particle in an electromagnetic field.

2. GENERAL PROPERTIES OF THE SPACES

The fundamentals of Finsler space are now widely available, so many explanatory details will be omitted.

The new class of Finsler metric function is defined as

$$F(x, y) = [(\eta_{\alpha\beta} + kB_\alpha B_\beta) y^\alpha y^\beta]^{1/2} \tag{1}$$

The vector $B_\alpha = B_\alpha(x, y)$ and k is some constant. In general the vector y^α is tangent to the point x^α. Often y^α is identified as the velocity vector $v^\alpha = dx^\alpha/d\tau$ along a timelike path with parameter τ.

The signature of $\eta_{\alpha\beta}$ is $(+1, -1, -1, -1)$. A more general Riemannian metric $g_{\alpha\beta}(x)$ could be inserted in place of $\eta_{\alpha\beta}$.

The function (1) can be compared with the well-known Randers form

$$F_R(x, y) = (g_{\alpha\beta} y^\alpha y^\beta)^{1/2} + k_\alpha y^\alpha \tag{2}$$

There are some similarities between the two, especially that both forms can produce equations which look something like the equations of classical charged particles. However, there are also fundamental differences; for example, the spaces determined by (1) are not in general C-reducible (Asanov, 1985, p. 76).

The standard homogeneity requirement for F

$$F(x, \lambda y) = \lambda F(x, y) \tag{3}$$

implies that B_α must be homogeneous of zero degree:

$$B_\alpha(x, \lambda y) = B_\alpha(x, y) \tag{4}$$

The requirements

$$(\partial F/\partial y^{\alpha})y^{\alpha} = F \tag{5}$$

and

$$(\partial^2 F/\partial y^{\alpha}\,\partial y^{\beta})y^{\alpha} = 0 \tag{6}$$

lead to

$$[\partial(B_{\alpha}y^{\alpha})/\partial y^{\beta}]y^{\beta} = B_{\alpha}y^{\alpha} \tag{7}$$

and

$$[\partial^2(B_{\nu}y^{\nu})/\partial y^{\alpha}\,\partial y^{\beta}]y^{\alpha} = 0 \tag{8}$$

Also, it is easy to show that

$$(\partial B_{\alpha}/\partial y^{\beta})y^{\beta} = 0 \tag{9}$$

The metric tensor for this class of space is

$$f_{\alpha\beta} = \tfrac{1}{2}(\partial^2 F^2/\partial y^{\alpha}\,\partial y^{\beta}) = \eta_{\alpha\beta} + k\tilde{B}_{\alpha}\tilde{B}_{\beta} + k(B_{\nu}y^{\nu})(\partial\tilde{B}_{\beta}/\partial y^{\alpha}) \tag{10}$$

The notation

$$\tilde{B}_{\beta} = \partial(B_{\alpha}y^{\alpha})/\partial y^{\beta} \tag{11}$$

has been introduced. From (7),

$$\tilde{B}_{\alpha}y^{\alpha} = B_{\alpha}y^{\alpha} \tag{12}$$

The relation

$$\partial\tilde{B}_{\alpha}/\partial y^{\beta} = \partial\tilde{B}_{\beta}/\partial y^{\alpha} \tag{13}$$

is obvious and verifies the symmetry of the metric.

Several connections can be defined in Finsler space. The one of concern here is the Christoeffel connection

$$\gamma_{\alpha\beta\nu} = \tfrac{1}{2}[(\partial f_{\alpha\nu}/\partial x^{\beta}) + (\partial f_{\nu\beta}/\partial x^{\alpha}) - (\partial f_{\alpha\beta}/\partial x^{\nu})] \tag{14}$$

For the present spaces

$$\begin{aligned}\gamma_{\alpha\beta\nu} = \tfrac{1}{2}k\{&[\partial\tilde{B}_{\alpha}/\partial x^{\beta}) + (\partial\tilde{B}_{\beta}/\partial x^{\alpha})]\tilde{B}_{\nu} + [(\partial\tilde{B}_{\nu}/\partial x^{\beta}) - (\partial\tilde{B}^{\beta}/\partial x^{\nu})]\tilde{B}_{\alpha}\\ &+ [(\partial\tilde{B}_{\nu}/\partial x^{\alpha}) - (\partial\tilde{B}_{\alpha}/\partial x^{\nu})]\tilde{B}_{\beta} + [\partial(B_{\mu}y^{\mu})/\partial x^{\alpha}]\,\partial\tilde{B}_{\nu}/\partial y^{\beta}\\ &+ [\partial(B_{\mu}y^{\mu})/\partial x^{\beta}]\,\partial\tilde{B}_{\nu}/\partial y^{\alpha} - [\partial(B_{\mu}y^{\mu})/\partial x^{\nu}]\,\partial\tilde{B}_{\beta}/\partial y^{\alpha}\\ &+ B_{\mu}y^{\mu}[(\partial^2\tilde{B}_{\nu}/\partial x^{\alpha}\,\partial y^{\beta}) + (\partial^2\tilde{B}_{\nu}/\partial x^{\beta}\,\partial y^{\alpha}) - (\partial^2\tilde{B}_{\alpha}/\partial x^{\nu}\,\partial y^{\beta})]\}\end{aligned} \tag{15}$$

The equation of a Finslerian geodesic for a timelike path is (Asanov, 1985)

$$dv^{\delta}/d\tau - v^{\delta}F^{-1}\,dF/d\tau + f^{\nu\delta}\gamma_{\alpha\beta\nu}v^{\alpha}v^{\beta} = 0 \tag{16}$$

The vector y^{α} has now been identified with the velocity v^{α}.

The contravariant metric tensor $f^{\nu\delta}$ is defined by

$$f^{\nu\delta}f_{\delta\varepsilon} = \delta^{\nu}_{\varepsilon} \tag{17}$$

For the general metric (10), $f^{\nu\delta}$ cannot be expressed in a closed form. However, there are many special cases with particular assumptions for B_{α} for which the contravariant metric can be given explicitly.

A simple example is $B_{\alpha} = B_{\alpha}(x)$. In this case $\tilde{B}_{\alpha} = B_{\alpha}$ and

$$f^{\nu\delta} = \eta^{\nu\delta} - k(1+kB^2)^{-1}B^{\nu}B^{\delta} \tag{18}$$

The notation $B^2 = B_{\alpha}B^{\alpha} = \eta_{\alpha\beta}B^{\alpha}B^{\beta}$ is used.

The geodesic equation becomes

$$\begin{aligned} &dv^{\delta}/d\tau - [c^2 + k(B_{\alpha}v^{\alpha})^2]^{-1}(B_{\nu}v^{\nu})v^{\delta}\,d(B_{\beta}v^{\beta})/d\tau \\ &\quad + k(1+kB^2)^{-1}v^{\alpha}B^{\delta}\,dB_{\alpha}/d\tau + B_{\beta}v^{\beta}f^{\nu\delta}H_{\alpha\nu}v^{\alpha} = 0 \\ &(H_{\alpha\nu} = \partial B_{\nu}/\partial x^{\alpha} - \partial B_{\alpha}/\partial x^{\nu}) \end{aligned} \tag{19}$$

As shown in Beil (1987), the geodesic equation can be rewritten as

$$\begin{aligned} &dv^{\delta}/d\tau + k[c^2 + k(B_{\alpha}v^{\alpha})^2]^{-1}[c^2B^{\delta} - (B_{\alpha}v^{\alpha})v^{\delta}]\,d(B_{\beta}v^{\beta})/d\tau \\ &\quad + k(B_{\beta}v^{\beta})\eta^{\nu\delta}H_{\alpha\nu}v^{\alpha} = 0 \end{aligned} \tag{20}$$

If a condition

$$B_{\alpha}v^{\alpha} = e/mck \tag{21}$$

can be imposed, then the geodesic equation has the form of the Lorentz equation. Also, the field equations for this metric produce a relation between k and the gravitational constant.

Of course, when B_{α} is not a function of v, then the space is no longer a Finsler space, but simply Riemannian. This Riemannian space could be considered a limiting case of this class of metrics.

To conclude this section, a more useful form of the geodesic equation (16) will be developed.

For the general metric (10), the intermediate results

$$\begin{aligned} \gamma_{\alpha\beta\nu}v^{\alpha}v^{\beta} &= k[\tilde{B}_{\nu}v^{\beta}\,\partial(B_{\alpha}v^{\alpha})/\partial x^{\beta} + B_{\beta}v^{\beta}\tilde{H}_{\alpha\nu}v^{\alpha}] \\ \tilde{H}_{\alpha\nu} &= \partial\tilde{B}_{\nu}/\partial x^{\alpha} - \partial\tilde{B}_{\alpha}/\partial x^{\nu} \end{aligned} \tag{22}$$

and

$$dF/d\tau = F^{-1}(\tfrac{1}{2}v^\alpha v^\beta \, df_{\alpha\beta}/d\tau + f_{\alpha\beta}v^\beta \, dv^\alpha/d\tau)$$
$$= F^{-1}(kB_\alpha v^\alpha v^\beta \, d\tilde{B}_\beta/d\tau + kB_\alpha v^\alpha \tilde{B}_\beta \, dv^\beta/d\tau) \tag{23}$$

are obtained.

When (22) and (23) are substituted into (16), the result is

$$dv^\delta/d\tau - kv^\delta B_\alpha v^\alpha F^{-2}(v^\beta \, d\tilde{B}_\beta/d\tau + \tilde{B}_\beta \, dv^\beta/d\tau)$$
$$+ kf^{\delta\nu}\tilde{B}_\nu v^\beta \, d\tilde{B}_\beta/d\tau + kB_\beta v^\beta f^{\delta\nu}\tilde{H}_{\alpha\nu}v^\alpha = 0 \tag{24}$$

where the relations

$$v^\beta \, \partial(B_\alpha v^\alpha)/\partial x^\beta = v^\alpha v^\beta \, \partial\tilde{B}_\alpha/\partial x^\beta = v^\alpha \, d\tilde{B}_\alpha/d\tau$$
$$= d(B_\alpha v^\alpha)/d\tau - \tilde{B}_\alpha \, dv^\alpha/d\tau \tag{25}$$

have been used.

The intent here is to remove the dependence on $dv^\delta/d\tau$ except in the leading term of (24). To this end, contract (24) with $\tilde{B}_\delta$ and solve for $\tilde{B}_\delta \, dv^\delta/d\tau$:

$$\tilde{B}_\delta \, dv^\delta/d\tau = [1 - kF^{-2}(B_\alpha v^\alpha)^2]^{-1}\{[kF^{-2}(B_\alpha v^\alpha)^2 - kf^{\alpha\beta}\tilde{B}_\alpha\tilde{B}_\beta]v^\beta \, d\tilde{B}_\beta/d\tau$$
$$-kB_\alpha v^\alpha f^{\delta\nu}\tilde{B}_\delta\tilde{H}_{\beta\nu}v^\beta\} \tag{26}$$

Equation (26 is then substituted into (24) and gives

$$dv^\delta/d\tau + k[f^{\delta\nu}\tilde{B}_\nu - c^{-2}B_\alpha v^\alpha(1 - kf^{\mu\nu}\tilde{B}_\mu\tilde{B}_\nu)v^\delta]v^\beta \, d\tilde{B}_\beta/d\tau$$
$$+kB_\lambda v^\lambda(f^{\delta\nu} + kc^{-2}B_\beta v^\beta f^{\mu\nu}\tilde{B}_\mu v^\delta)\tilde{H}_{\alpha\nu}v^\alpha = 0 \tag{27}$$

This version of the geodesic equation is valid for any metric of the class (10).

3. EXAMPLES OF PARTICULAR SPACES

Some of the simpler possible choices for B_α are listed:

$$\begin{aligned} B^1_\alpha &= (s_\beta v^\beta)^{-1}v_\alpha \\ B^2_\alpha &= (s_\beta v^\beta)^{-1}A_\nu v^\nu s_\alpha \\ B^3_\alpha &= (v_\beta v^\beta)^{1/2}(s_\nu v^\nu)^{-1}s_\alpha \\ B^4_\alpha &= v_\beta v^\beta(s_\nu v^\nu)^{-2}s_\alpha \\ B^5_\alpha &= (v_\beta v^\beta)^{-1/2}v_\alpha \\ B^6_\alpha &= A_\beta v^\beta(s_\nu v^\nu)^{-2}v_\alpha \end{aligned} \tag{28}$$

Here $A_\alpha(x)$ and $s_\alpha(x)$ are vectors to be specified. If v^α is just the velocity vector, then $v_\alpha v^\alpha = c^2$ can be inserted.

The selection from the above, or the other possible choices for B_α, will be determined by the physical system being modeled. It is only required that B_α be of zero degree homogeneity.

The metric for each of these B_α can be readily obtained from (10). For example, for B^1_α,

$$f^1_{\alpha\beta}=[1+2kv_\nu v^\nu(s_\beta v^\beta)^{-2}]\eta_{\alpha\beta}-4kv_\nu v^\nu(s_\beta v^\beta)^{-3}(s_\alpha v_\beta+s_\beta v_\alpha)$$
$$+4kv_\alpha v_\beta(s_\nu v^\nu)^{-2}+3k(v_\nu v^\nu)^2(s_\mu v^\mu)^{-4}s_\alpha s_\beta \tag{29}$$

When the metric is computed for some of these B_α the space is found to actually be Riemannian. For example,

$$f^2_{\alpha\beta}=\eta_{\alpha\beta}+kA_\alpha a_\beta \tag{30}$$

which is equivalent to the limiting case $B_\alpha=B_\alpha(x)$.

B^3_α also produces a Riemannian space since

$$f^3_{\alpha\beta}=(1+k)\eta_{\alpha\beta} \tag{31}$$

Metrics can also be constructed by taking B_α to be a linear combination of any of the above choices. Interestingly, linear combinations such as

$$B^e_\nu=B^2_\nu+\alpha B^3_\nu \qquad (\alpha \text{ a constant}) \tag{32}$$

do not reduce to Riemannian spaces. This metric is

$$\begin{aligned} f^e_{\alpha\beta}&=[1+k\alpha B^e_\nu v^\nu(v_\mu v^\nu)^{-1/2}]\eta_{\alpha\beta}+k\tilde{B}^e_\alpha\tilde{B}^e_\beta-k\alpha B^e_\nu v^\nu(v_\mu v^\nu)^{-3/2}v_\alpha v_\beta \\ \tilde{B}^e_\alpha&=A_\alpha+\alpha(v_\nu v^\nu)^{-1/2}v_\alpha \end{aligned} \tag{33}$$

A computational difficulty in working with these spaces is the determination of the contravariant form of the metric. For a metric such as (29) this can be a tedious task. However, a general result has been obtained which can save considerable labor: For a general metric

$$f_{\alpha\beta}=\alpha_1\eta_{\alpha\beta}+\alpha_2 a_\alpha a_\beta+\alpha_3(a_\alpha b_\beta+b_\alpha a_\beta)+\alpha_4 b_\alpha b_\beta \tag{34}$$

The contravariant form is

$$f^{\beta\nu}=\beta_1\eta^{\beta\nu}+\beta_2 a^\beta a^\nu+\beta_3(a^\beta b^\nu+b^\beta a^\nu)+\beta_4 b^\beta b^\nu \tag{35}$$

with

$$\begin{aligned} \beta_1&=\alpha_1^{-1} \\ \beta_2&=\alpha_1^{-1}\mathscr{A}^{-1}[\alpha_1\alpha_2+(\alpha_2\alpha_4-\alpha_3^2)b_\nu b^\nu] \\ \beta_3&=\alpha_1^{-1}\mathscr{A}^{-1}[\alpha_1\alpha_3-(\alpha_2\alpha_4-\alpha_3^2)a_\nu b^\nu] \\ \beta_4&=\alpha_1^{-1}\mathscr{A}^{-1}[\alpha_1\alpha_4+(\alpha_2\alpha_4-\alpha_3^2)a_\nu a^\nu] \\ \mathscr{A}&=(\alpha_2\alpha_4-\alpha_3^2)[(a_\nu b^\nu)^2-(a_\nu a^\nu)(b_\mu b^\mu)] \\ &\quad -\alpha_1[\alpha_1+\alpha_2(a_\nu a^\nu)+2\alpha_3(a_\nu b^\nu)+\alpha_4(b_\nu b^\nu)] \end{aligned} \tag{36}$$

This satisfies (17) as required.

A metric is now selected for detailed analysis in order to illustrate some possibilities of this class of Finsler space. The metric is the one given by (33), which is obtained from (32) or by taking

$$B^e_\mu = A_\mu + \alpha(v_\nu v^\nu)^{1/2}(s_\beta v^\beta)^{-1} s_\mu$$

This is considered to be a special case of (34) with $a_\alpha = v_\alpha$, $b_\alpha = \tilde{B}^e_\alpha = A_\alpha + \alpha(v_\nu v^\nu)^{-1/2} v_\alpha$. Then,

$$\begin{aligned} \beta_1 &= 1 + k\alpha c^{-1} B^e_\nu v^\nu \\ \beta_2 &= -k\alpha c^{-3} B^e_\nu v^\nu \\ \beta_3 &= 0 \\ \beta_4 &= k \\ \mathcal{A} &= -[\beta_1 + k\tilde{B}^e_\nu \tilde{B}^{e\nu} + k^2\alpha c^{-3}(B^e_\nu v^\nu)^3] \end{aligned} \tag{37}$$

So,

$$\begin{aligned} f^{\beta\nu} = \beta_1^{-1}\{\eta^{\beta\nu} - \mathcal{A}^{-1}[k\alpha c^{-3} B_\alpha v^\alpha(\beta_1 + k\tilde{B}_\mu \tilde{B}^\mu) v^\beta v^\nu \\ -k^2\alpha c^{-3}(B_\alpha v^\alpha)^2(v^\beta \tilde{B}^\nu + \tilde{B}^\beta v^\nu) - k\tilde{B}^\beta \tilde{B}^\nu]\} \end{aligned} \tag{38}$$

The superscript e will be omitted from B^e for the rest of this section.

The geodesic equation for this special case will be obtained using (27). A significant result is

$$\tilde{H}_{\alpha\nu} = \partial A_\nu/\partial x^\alpha - \partial A_\alpha/\partial x^\nu = F_{\alpha\nu} \tag{39}$$

The vector A_ν could be identified with the electromagnetic potential vector. Equation (39) demonstrates that a term in B_α which is just A_α or B^2_α behaves just like the potential vector in the quation of motion. Further, terms in B_α which become purely velocity dependent in $\tilde{B}_\alpha$ (e.g., B^3_α or B^5_α) behave like electromagnetic gauge transformations when added to A_α terms.

Other intermediate results for parts of the geodesic equation are

$$f^{\delta\nu}\tilde{B}_\nu = -\mathcal{A}^{-1}[k\alpha c^{-3}(B_\nu v^\nu)^2 v^\delta + \tilde{B}^\delta] \tag{40}$$

and

$$f^{\delta\nu}\tilde{B}_\delta \tilde{B}_\nu = -\mathcal{A}^{-1}[k\alpha c^{-3}(B_\nu v^\nu)^3 + \tilde{B}_\gamma \tilde{B}^\nu] \tag{41}$$

Recall that the indices of v and $\tilde{B}$ are raised and lowered by $\eta_{\alpha\beta}$.

When (38)-(41) are used in (27) the result can be shown to be

$$\begin{aligned} dv^\delta/d\tau + k\mathcal{A}^{-1}(c^{-2}A_\nu v^\nu v^\delta - A^\delta) v^\alpha\, d\tilde{B}_\alpha/d\tau \\ + k\beta_1^{-1} B_\nu v^\nu[\eta^{\delta\nu} - k\mathcal{A}^{-1}(c^{-2}A_\nu v^\nu v^\delta - A^\delta)\tilde{B}^\nu] F_{\alpha\nu} v^\alpha = 0 \end{aligned} \tag{42}$$

It is tentatively assumed that the factor $k\mathscr{A}^{-1}$ is small. This means that (42) is equivalent to the Lorentz equation if

$$kB_\nu v^\nu \beta_1^{-1} = (1 + k\alpha c^{-1} A_\nu v^\nu + k\alpha^2)^{-1}(kA_\alpha v^\alpha + k\alpha c) = e/mc \qquad (43)$$

The relative size of terms is determined by α if k is of "moderate" size (i.e., $10^{-6} < k < 10^6$ in cgs units), and A^μ not too large. Thus,

$$\alpha = mc^2/e \approx 10^6 \quad (\text{g cm/sec}^2)^{1/2} \qquad (44)$$

to a good approximation. The magnitude of $\mathscr{A}$ is then determined by the term $k^2\alpha^4$, which must be greater than 10^{12}. This confirms the assumption that $k\mathscr{A}^{-1}$ is small.

Thus, a particular choice for B_α produces a geodesic equation of some physical significance. Other choices to model other physcial situations are possible.

4. DISCUSSION

The first objection (i) to the physical application of Finsler spaces is based on the fact that a general null geodesic condition

$$f_{\alpha\beta}(x, y) y^\alpha y^\beta = 0 \qquad (45)$$

may have multiple solutions for y and that these solutions for $y = dx$ may not be light cones.

However, as pointed out by Ishikawa (1981), there is an alternate definition of a null geodesic which is more reasonable from the standpoint of physical causality:

$$f_{\alpha\beta}(x, y)\, dx^\alpha\, dx^\beta = 0 \qquad (46)$$

Here the Finsler metric is defined in terms of a vector field y from some reference frame and y is not related to dx, the null cone coordinates. The light cone is always observed in a particular frame defined by y. This how all such measurements are done.

In (46) y is treated like a parameter, so the solution for dx is unique and is the usual light cone as would be obtained in a Riemannian space from some $g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta = 0$. The vector y may be identified with the frame velocity v.

The second objection (ii) relates to the definition of the covariant derivative. There are two definitions commonly used in Finsler space, the δ derivative of Rund and the Cartan derivative. These are usually indicated by the subscripts ; and |, respectively. See Asanov (1985) for precise definitions.

The Cartan derivative is specifically defined so that the length or norm of vectors remains invariant under parallel transport (Rund, 1959). This is equivalent to

$$f_{\alpha\beta|\nu}=0 \tag{47}$$

which is the same as the metric property of the covariant derivative in Riemannian space. So the objection (ii) does not apply to the Cartan derivative.

The Cartan derivative is applicable to spaces where x and y are independent variables. This is the type of space considered in the present paper. However, it is also possible that the tangent vector y can be a function of x. For these spaces, with $y=y(x)$, the δ derivative must be used, which no longer has the metric property. That is,

$$f_{\alpha\beta;\nu}=y^{\mu}_{;\nu}\,\partial f_{\alpha\beta}/\partial y^{\mu} \tag{48}$$

for the derivative along the y direction. This is not as serious as it appears, however, since the differential of $f_{\alpha\beta}$ does vanish along geodesics. The norms of vectors then are invariant along geodesic paths.

So (ii) does not apply to most cases of physical interest; indeed, it is only valid for nongeodesic paths where $y=y(x)$.

In view of the above, Finsler spaces should be given more careful consideration as a generalization of the Riemannian approach to general relativity. As shown here, they can reproduce equations of physical interest, but from new points of view which might allow extensions of theory in new directions.

ACKNOWLEDGMENT

I thank Prof. G. S. Asanov for helpful comments and encouragement in this work.

REFERENCES

Adler, R., Bazin, M., and Schiffer, M. (1975). *Introduction to General Relativity*, McGraw-Hill, New York.

Aringazin, A. K., and Asanov, G. S. (1985). *General Relativity and Gravitation*, **17**, 1153.

Asanov, G. S. (1977). *Reports on Mathematical Physis*, **11**, 221.

Asanov, G. S. (1985). *Finsler Geometry, Relativity, and Gauge Theories*, D. Reidel, Dordrecht, Holland.

Beil, R. G. (1987). *International Journal of Theoretical Physics*, **26**, 189.

Ikeda, S. (1985). *Journal of Mathematical Physics*, **26**, 958.

Ishikawa, H. (1981). *Journal of Mathematical Physics*, **22**, 995.

Rund, H. (1959). *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin.

Tavakol, R. K., and Van den Bergh, N. (1986). *General Relativity and Gravitation*, **18**, 849.